

# LYUBEZNIK NUMBERS IN MIXED CHARACTERISTIC

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**ABSTRACT.** This manuscript defines a new family of invariants, analogous to the *Lyubeznik numbers*, associated to *any* local ring whose residue field has prime characteristic. In particular, as their nomenclature suggests, these *Lyubeznik numbers in mixed characteristic* are defined for all local rings of mixed characteristic. Some properties similar to those in equal characteristic hold for these new invariants. Notably, the “highest” Lyubeznik number in mixed characteristic is a well-defined notion. Although the Lyubeznik numbers in mixed characteristic and their equal-characteristic counterparts are the same for certain local rings of equal characteristic  $p > 0$ , we also provide an example where they differ.

## 1. INTRODUCTION

**1.1. Background.** Let  $I$  be an ideal of a regular local ring  $S$  of equal characteristic. When  $S$  has prime characteristic, Huneke and Sharp proved that the Bass numbers of every local cohomology module of the form  $H_I^i(S)$  are finite [HS93]. Subsequently, Lyubeznik showed this for  $S$  of equal characteristic zero using  $D$ -module theory. He also extended these finiteness results to a larger family of functors that include iterated local cohomology modules of the form  $H_{I_1}^{i_1} \cdots H_{I_\ell}^{i_\ell}(S)$ , for  $I_1, \dots, I_\ell$  ideals of  $S$  and  $i_1, \dots, i_\ell \in \mathbb{N}$ . Using these results, he defined families of invariants that are now called *Lyubeznik numbers* [Lyu93].

Suppose that  $(R, m, K)$  is a local ring admitting a surjection from an  $n$ -dimensional local regular local ring  $(S, \eta, K)$  containing a field. If  $I$  is the kernel of this surjection, the Lyubeznik numbers of  $R$ , depending on two nonnegative integers  $i$  and  $j$ , are defined as  $\lambda_{i,j}(R) := \dim_K \operatorname{Ext}_S^i(K, H_I^{n-j}(S))$ . Remarkably, these numbers only depend on the ring  $R$  and on  $i$  and  $j$ , not on  $S$ , nor even on the choice of surjection from  $S$ . Moreover, if  $R$  is *any* local ring containing a field, letting  $\lambda_{i,j}(R) := \lambda_{i,j}(\widehat{R})$  extends the definition, making the Lyubeznik numbers well defined for every such ring [Lyu93].

For  $R$  containing a field, the Lyubeznik numbers of  $R$  provide essential information about the ring. For example, if  $d = \dim R$ , then  $\lambda_{i,j}(R) = 0$  for  $j > d$ , and  $\lambda_{d,d}(R) \neq 0$  [Lyu93, Properties 4.4i, 4.4iii]. If  $R$  is a Cohen-Macaulay ring, then  $\lambda_{d,d}(R) = 1$  [Kaw02, Theorem 1]. Moreover, the Lyubeznik numbers have extensive connections with geometry and topology, including étale cohomology and the connected components of certain punctured spectra. (See, for example, [BB05, GLS98, Kaw00, Wal01, Zha07].)

**1.2. Aim and Main Results.** The aim of this manuscript is to define a new invariant, analogous to the Lyubeznik numbers, which are associated to *any* local ring whose residue field has prime characteristic. In particular, these new invariants are defined for local rings of mixed characteristic. Moreover, we study properties of the

Lyubeznik numbers in mixed characteristic, and investigate when they agree with the equal-characteristic Lyubeznik numbers, as well as whether they ever differ from them.

If  $S$  is a regular local ring of unramified mixed characteristic, then Lyubeznik, along with Núñez-Betancourt, have shown that the Bass numbers are also finite [Lyu00b, NB12]. This enables us to give the following definition:

**Proposition/Definition (3.4, 3.5).** Let  $(R, m, K)$  be a local ring such that  $\text{char}(K) = p > 0$ , and let  $\widehat{R}$  denote its completion. By the Cohen Structure Theorems,  $\widehat{R}$  admits a surjective ring homomorphism  $\pi : S \rightarrow \widehat{R}$ , where  $S$  is an unramified regular local ring of mixed characteristic. Let  $I = \text{Ker}(\pi)$  and  $n = \dim(S)$ . Then, given integers  $i, j \geq 0$ ,  $\dim_K \text{Ext}_S^i(K, H_I^{n-j}(S))$  is finite and depends only on  $R, i$ , and  $j$ . We define the *Lyubeznik numbers in mixed characteristic of  $R$  with respect to  $i$  and  $j$*  as

$$\widetilde{\lambda}_{i,j}(R) := \dim_K \text{Ext}_S^i(K, H_I^{n-j}(S)).$$

When  $R$  is a ring of equal characteristic  $p > 0$ , both the Lyubeznik numbers and the Lyubeznik numbers in mixed characteristic are defined. We give some conditions for which these two definitions agree. However, in Theorem 6.10, we prove that these numbers are not the same in general.

Motivated by analogous properties of the Lyubeznik numbers in equal characteristic, we study properties of these invariants (cf. [Lyu93, Properties 4.4]). Some similar vanishing properties hold (see Proposition 3.9), as well as analogous computations for complete intersection rings (see Proposition 3.10). Moreover, the “highest” Lyubeznik number in mixed characteristic of a local ring for which these invariants are defined is a well-defined notion: if  $(R, m, K)$  is a local ring of dimension  $d$  such that  $\text{char}(R) = p > 0$ ,  $\widetilde{\lambda}_{i,j}(R) = 0$  if either  $i > d$  or  $j > d$  (see Theorem 4.10 and Definition 4.11).

We investigate conditions on a local ring  $R$  of equal characteristic  $p > 0$  (i.e., when both the Lyubeznik numbers and the Lyubeznik numbers in mixed characteristic of  $R$  are defined) under which  $\widetilde{\lambda}_{i,j}(R) = \lambda_{i,j}(R)$  for all  $i, j \in \mathbb{N}$ . We find that they agree for  $R$  Cohen-Macaulay, and  $R$  of dimension less than or equal to two (see Corollary 5.4).

Moreover, we give a specific example for which  $\widetilde{\lambda}_{i,j}(R) \neq \lambda_{i,j}(R)$  for some  $i, j \in \mathbb{N}$ , employing the work on Bockstein morphisms of Singh and Walther, as well as a computation of Álvarez Montaner and Vahidi [SW11, ÀMV] (see Remark 6.9 and Theorem 6.10).

**1.3. Outline.** We recall some definitions and properties of  $D$ -modules in Section 2. In addition, we introduce a functor that will play a fundamental role in proving that the Lyubeznik numbers in mixed characteristic are well defined. In Section 3, we define the Lyubeznik numbers in mixed characteristic, and prove some similar properties to those that hold for the Lyubeznik numbers in equal characteristic. Next, in Section 4, we prove in Theorem 4.10 that, with our definition, if  $d = \dim R$ ,  $\widetilde{\lambda}_{d,d}(R)$  is again the “highest” Lyubeznik number. The goal of Section 5 is to prove that the original Lyubeznik numbers and Lyubeznik numbers in mixed characteristic are the same for certain rings of characteristic  $p > 0$ . On the other hand, in Section 6, we give an example where these numbers differ.

## 2. PRELIMINARIES

**2.1.  $D$ -modules.** Here, we briefly review some facts about  $D$ -modules that will be used; we refer readers to [Bjö72, Cou95] for a more detailed exposition.

If  $R \subseteq S$  is an extension of commutative rings, the  $R$ -linear differential operators of  $S$ , denoted  $D(S, R)$ , is the (non-commutative) subring of  $\text{End}_R(S)$  defined inductively as follows:

- The operators of order zero are the endomorphisms defined by multiplication by some element of  $S$ , and
- $\phi \in \text{End}_R(S)$  is an operator of order less than or equal to  $\ell$  if  $[\phi, \sigma] = \phi \cdot \sigma - \sigma \cdot \phi$  is an operator of order less than or equal to  $\ell - 1$  for every  $\sigma \in S$ .

If  $S = R[[x_1, \dots, x_n]]$  for some  $n \in \mathbb{N}$ , then

$$D(S, R) = S \left\langle \frac{1}{t!} \frac{\partial^t}{\partial x_i^t} \mid t \in \mathbb{N}, i = 1, \dots, n \right\rangle \subseteq \text{Hom}_R(S, S)$$

[Gro67, Theorem 16.12.1].

The canonical map  $M \rightarrow M_f$  is a map of  $D(S, R)$ -modules for any  $D(S, R)$ -module  $M$ , and any  $f \in S$ . Since  $S$  is a  $D(S, R)$ -module, this map induces a natural  $D(S, R)$ -module structure on any local cohomology module of the form  $H_I^j(S)$ , where  $I$  is an ideal of  $S$  and  $j \in \mathbb{N}$ . Moreover, it is not hard to see that this also makes  $H_{I_s}^{j_\ell} \dots H_{I_2}^{j_2} H_{I_1}^{j_1}(S)$  a  $D(S, R)$ -module as well [Lyu93, Example 2.1(iv)]. Given a field  $K$  and any  $f \in S$ ,  $S_f$  finite length in the category of  $D(S, K)$ -modules. Thus,  $H_{I_s}^{j_\ell} \dots H_{I_2}^{j_2} H_{I_1}^{j_1}(S)$  does as well [Lyu00a, Corollary 6].

**2.2. A key functor.** Lyubeznik introduced a certain functor in proving that the equal-characteristic Lyubeznik numbers are well defined (cf. [Lyu93, Lemma 4.3]). In [NBW12, Section 2.2], the authors study further properties of this functor. As we will rely greatly on some of these properties to prove that the Lyubeznik numbers in mixed characteristic are well defined, we include them here for the reader's convenience. Please refer to *loc. cit.* for all details and proofs.

Fix a Noetherian ring  $R$ , and let  $S = R[[x]]$ . Let  $G$  denote the functor from the category of  $R$ -modules to that of  $S$ -modules given by  $G(-) = (-) \otimes_R S_x/S$ . Fix an  $R$ -module  $M$ . Since  $G(M) = M \otimes_R S_x/S = \bigoplus_{\alpha \in \mathbb{N}} (M \otimes R x^{-\alpha})$ , any  $u \in G(M)$  can be

written uniquely in the form  $u = m_1 \otimes x^{-\alpha_1} + \dots + m_\ell \otimes x^{-\alpha_\ell}$  for some  $\ell, \alpha_i, m_i \in \mathbb{N}$ .

Actually,  $G$  is an equivalence between the category of  $R$ -modules to that of  $D(S, R)$ -modules supported on the  $\mathcal{V}(xS)$ , the Zariski closed subset of  $\text{Spec}(S)$  given by  $xS$ . The  $D(S, R)$ -module structure of  $G(M)$  is defined by the action of each  $\frac{1}{t!} \frac{\partial^t}{\partial x^t}$ ,  $t \in \mathbb{N}$ :  $\left(\frac{1}{t!} \frac{\partial^t}{\partial x^t}\right) \cdot (m \otimes x^{-\alpha}) = \binom{\alpha+t-1}{t} \cdot ((-1)^t m \otimes x^{-\alpha-t})$ . The functor  $G$  is also flat, and commutes with taking local cohomology.

For every ideal  $I$  of  $R$  and every  $j \in \mathbb{N}$ ,  $G(H_I^j(M)) \cong H_{(I,x)S}^{j+1}(M \otimes_R S)$ . Moreover, if  $R$  is Gorenstein,  $P$  is a prime ideal of  $R$ , and  $E_R(R/P)$  denotes the injective hull of  $R/P$  over  $R$ ,  $G(E_R(R/P)) = E_S(S/(P, x)S)$ . In fact, in this case,  $M$  is an injective  $R$ -module if and only if  $G(M)$  is an injective  $S$ -module. Using this fact, it can also be shown that, endowing  $R = S/xS$  with the natural  $S$ -module structure via extension of scalars, if  $N$  is also an  $R$ -module,  $\text{Ext}_S^j(M, G(N)) = \text{Ext}_R^j(M, N)$  for all  $j \in \mathbb{N}$ .

## 3. DEFINITION AND PROPERTIES

**Lemma 3.1.** *Let  $S = R[[x]]$ , where  $(R, m, K)$  is a Gorenstein local ring. Then for every ideal  $I$  of  $R$ , and all  $i, j \in \mathbb{N}$ ,*

$$\dim_K \operatorname{Ext}_S^i(K, H_{(I,x)S}^{j+1}(S)) = \dim_K \operatorname{Ext}_R^i(K, H_I^j(R)).$$

*Proof.* Let  $G$  be the functor defined in Section 2.2. Now,  $G(H_I^j(R)) = H_{(I,x)}^{j+1}(S)$  by [NBW12, Lemma 3.9]. Since  $R$  is Gorenstein, [NBW12, Proposition 3.12] indicates that  $\operatorname{Ext}_S^i(M, G(N)) = \operatorname{Ext}_R^i(M, N)$  for all  $R$ -modules  $M$  and  $N$ . Thus,  $\operatorname{Ext}_S^i(K, H_{(I,x)S}^{j+1}(S)) = \operatorname{Ext}_R^i(K, H_I^j(R))$ , and we are done.  $\square$

**Corollary 3.2.** *Let  $(R, m, K)$  be a Gorenstein local ring, and let  $S = R[[x_1, \dots, x_n]]$  denote a power series ring over  $R$ . For every ideal  $I$  of  $R$  and all  $i, j \in \mathbb{N}$ ,*

$$\dim_K \operatorname{Ext}_S^i(K, H_{(I,x_1,\dots,x_n)S}^{j+n}(S)) = \dim_K \operatorname{Ext}_R^i(K, H_I^j(R)).$$

*Proof.* Using Lemma 3.1, apply induction on  $n$ .  $\square$

**Remark 3.3** ([Coh46]). If  $(V, pV, K)$  and  $(W, pW, K)$  are both complete Noetherian DVRs with residue class field  $K$ , then  $V$  and  $W$  are isomorphic (via a non-unique isomorphism).

By the Cohen Structure Theorems, if  $(R, m, K)$  is a complete local ring of mixed characteristic  $p > 0$ ,  $R$  is the homomorphic image of a ring of the form  $V[[x_1, \dots, x_n]]$ , where  $(V, pV, K)$  is an (unramified) mixed characteristic complete Noetherian discrete valuation domain. Therefore, if  $V[[x_1, \dots, x_n]] \twoheadrightarrow R$  and  $W[[y_1, \dots, y_{n'}]] \twoheadrightarrow R$ , then  $V \cong W$ . (In fact, we can take  $n = n'$  to be the embedding dimension of  $R/pR$ .)

If  $(R, m, K)$  is a complete local ring of equal characteristic  $p > 0$ ,  $R$  is a homomorphic image of some  $K[[x_1, \dots, x_n]]$  by the Cohen Structure Theorems. Therefore, if  $(V, pV, K)$  is the complete Noetherian DVR provided above, the composition  $V[[x_1, \dots, x_n]] \twoheadrightarrow K[[x_1, \dots, x_n]] \twoheadrightarrow R$  is a surjection.

Thus, any complete local ring  $(R, m, K)$  such that  $\operatorname{char}(K) = p > 0$  is the homomorphic image of  $V[[x_1, \dots, x_n]]$ , where  $(V, pV, K)$  is a uniquely determined mixed characteristic complete Noetherian discrete valuation domain.

**Proposition 3.4.** *Let  $(R, m, K)$  be a local ring such that  $\operatorname{char}(K) = p > 0$ , admitting a surjection  $\pi : S \twoheadrightarrow R$ , where  $S$  is an unramified regular local ring of dimension  $n$ . Let  $I = \operatorname{Ker}(\pi)$ , and take  $i, j \in \mathbb{N}$ . Then*

$$\dim_K \operatorname{Ext}_S^{n-i}(K, H_I^j(S))$$

*is finite and depends only on  $R$ ,  $i$ , and  $j$ , but not on  $S$ , nor on  $\pi$ .*

*Proof.* Each  $\dim_K \operatorname{Ext}_S^i(K, H_I^{n-j}(S))$  is finite (cf. [Lyu00b, NB12]), so it remains to prove that these numbers are well defined. Let  $\pi' : S' \twoheadrightarrow R$  be another surjective map, where  $S'$  is an unramified regular local ring of dimension  $n'$ . Set  $I' = \operatorname{Ker}(\pi')$ , and let  $m'$  be the maximal ideal of  $S'$ . Since the Bass numbers with respect to the maximal ideal are not affected by completion, we may assume that  $R$ ,  $S$ , and  $S'$  are complete. Let  $V$  denote the complete DVR associated to  $K$  given by the Cohen Structure Theorems (see Remark 3.3), so that we can make identifications  $S = V[[x_1, \dots, x_{n-1}]]$  and  $S' = V[[y_1, \dots, y_{n'-1}]]$ .

Let  $S'' = V[[z_1, \dots, z_{n+n'-2}]]$ , and let  $\pi'' : S'' \twoheadrightarrow R$  be the surjective map defined by  $\pi''(z_j) = \pi(x_j)$  for  $1 \leq j \leq n-1$  and  $\pi''(z_j) = \pi'(y_{j-n+1})$  for  $n \leq j \leq n+n'-2$ . Let  $I''$  be the preimage of  $I$  under  $\pi''$ , respectively. Let  $\alpha : S \rightarrow S''$  be the map defined by  $\alpha(x_j) = z_j$ . Note that  $\pi'' \circ \alpha = \pi$ . Since  $\pi'$  is surjective, there exist  $f_1, \dots, f_{n'-1} \in S$  such that  $\pi''(z_{n-1+j}) = \pi(f_j)$  for  $j \leq n'-1$ . Then  $z_{n-1+j} - \alpha(f_j) \in \text{Ker}(\pi'')$ . We note that  $\beta : S'' \rightarrow S$ , defined by sending  $z_j$  to  $x_j$  for  $j \leq n-1$ , and  $z_{n-1+j}$  to  $f_j$  for  $j \leq n'-1$ , is a splitting of  $\alpha$ . Then  $I'' = (I, z_n - \alpha(f_1), \dots, z_{n'+n-2} - \alpha(f_{n'-1}))S''$ . Since

$$z_1, \dots, z_{n-1}, z_n - \alpha(f_1), \dots, z_{n'+n-2} - \alpha(f_{n'-1})$$

form a regular system of parameters, we obtain, by Corollary 3.2, that

$$\dim_K \text{Ext}_{S''}^i(K, H_{I''}^{n+n'-j}(S'')) = \dim_K \text{Ext}_S^i(K, H_I^{n-j}(S)).$$

By an analogous argument, we also have that

$$\dim_K \text{Ext}_{S''}^i(K, H_{I''}^{n+n'-j}(S'')) = \dim_K \text{Ext}_{S'}^i(K, H_{I'}^{n'-j}(S')),$$

and the result follows.  $\square$

**Definition 3.5** (Lyubeznik numbers in mixed characteristic). Let  $(R, m, K)$  be a local ring such that  $\text{char}(K) = p > 0$ . By the Cohen Structure Theorems, the completion  $\widehat{R}$  admits a surjection  $\pi : S \twoheadrightarrow \widehat{R}$ , where  $S$  is an unramified regular local ring of mixed characteristic. Let  $I = \text{Ker}(\pi)$ ,  $n = \dim(S)$ , and  $i, j \in \mathbb{N}$ . We define the *Lyubeznik number in mixed characteristic of  $R$  with respect to  $i$  and  $j$*  as

$$\widetilde{\lambda}_{i,j}(R) := \dim_K \text{Ext}_S^i(K, H_I^{n-j}(S)).$$

Note that by Proposition 3.4, the  $\widetilde{\lambda}_{i,j}(R)$  are well defined and depend only on  $R$ ,  $i$ , and  $j$ .

**Remark 3.6.** In Definition 3.5, we need to take the completion of  $R$  for the Cohen Structure Theorems to ensure the existence of a surjection from an unramified regular local ring  $S$  of mixed characteristic,  $\pi : S \twoheadrightarrow \widehat{R}$ . If such a map exists without taking the completion, then  $\widetilde{\lambda}_{i,j}(R) = \dim_K \text{Ext}_{\widehat{S}}^i(K, H_{\widehat{I}}^{n-j}(\widehat{S})) = \dim_K \text{Ext}_S^i(K, H_I^{n-j}(S))$ , where  $I = \text{Ker}(\pi)$ .

**Remark 3.7.** Fix  $(R, m, K)$ , a local ring of equal characteristic  $p > 0$ . On one hand, there exists a surjection from an  $n$ -dimensional unramified regular local ring of mixed characteristic,  $\pi : S \twoheadrightarrow \widehat{R}$ . On the other hand, the induced map  $\pi' : S/pS \twoheadrightarrow \widehat{R}$  is also surjective. If  $I = \text{Ker}(\pi)$  and  $I' = \text{Ker}(\pi')$ ,  $I$  is the preimage of  $I'$  under the canonical surjection  $S \twoheadrightarrow S/pS$ . There are two notions of Lyubeznik numbers corresponding to these homomorphisms, those given by Lyubeznik's original definition for rings of equal characteristic,  $\lambda_{i,j}(R) = \dim_K \text{Ext}_{S/pS}^i(K, H_{I'}^{n-j-1}(S/pS))$  [Lyu93], and the Lyubeznik numbers in mixed characteristic,  $\widetilde{\lambda}_{i,j}(R) = \dim_K \text{Ext}_S^i(K, H_I^{n-j}(S))$ .

Remark 3.7 naturally incites the following question:

**Question 3.8.** Is  $\widetilde{\lambda}_{i,j}(R) = \lambda_{i,j}(R)$  whenever both are defined, i.e., for every local ring  $(R, m, K)$  of equal characteristic  $p > 0$  and all  $i, j \in \mathbb{N}$ ? In other words, with notation as in Remark 3.7, is it always true that  $\dim_K \text{Ext}_{S/pS}^i(K, H_{I'}^{n-j-1}(S/pS)) = \dim_K \text{Ext}_S^i(K, H_I^{n-j}(S))$ ?

In Corollary 5.4, we prove an affirmative answer to Question 3.8 for certain rings. However, there are cases in which the answer is negative: Theorem 6.10 presents such an example, a Stanley-Reisner ring over a field of characteristic two.

The Lyubeznik numbers in mixed characteristic satisfy some similar vanishing properties to those that hold for the equal characteristic ones:

**Proposition 3.9.** *Let  $(R, m, K)$  be a local ring such that  $\text{char}(K) = p > 0$  and  $d = \dim(R)$ . Then*

- (i)  $\tilde{\lambda}_{i,j}(R) = 0$  for  $j > d$ ,
- (ii)  $\tilde{\lambda}_{i,j}(R) = 0$  for  $i > j + 1$ , and
- (iii)  $\tilde{\lambda}_{d,d}(R) \neq 0$ .

*Proof.* Let  $\hat{R}$  be the completion of  $R$ , so that  $\hat{R}$  admits a surjective ring homomorphism  $\pi : S \rightarrow \hat{R}$ , where  $(S, \eta)$  is an unramified regular local ring of mixed characteristic and of dimension  $n$ . Let  $I = \text{Ker}(\pi)$ .

Property (i) holds since  $H_I^{n-j}(S) = 0$  for  $j > \dim(S/I) = \dim R = d$ , and (ii) holds since  $\text{inj. dim}_S H_I^{n-j}(S) \leq \dim_S H_I^{n-j}(S) + 1 \leq i + 1$  [Zho98].

To prove (iii), first note that by an analogous argument to the proof of [Lyu93, Property 4.4(iii)],  $H_\eta^d H_I^{n-d}(S) \neq 0$ . We will prove that  $\tilde{\lambda}_{d,d}(R) \neq 0$  by contradicting this fact. Suppose that  $\tilde{\lambda}_{d,d}(R) = \text{Ext}_S^d(K, H_I^{n-d}(S)) = 0$ .

We claim that  $\text{Ext}_S^d(M, H_I^{n-d}(S)) = 0$  for every finite-length  $S$ -module  $M$ . We will prove this by induction on  $h = \text{length}_S(M)$ . If  $h = 1$ , then  $M = K$ , and the statement holds by assumption. Suppose that the statement is true for all  $N$  with  $\text{length}_S N < h + 1$ , and take  $M$  with  $\text{length}_S M = h + 1$ . Then there exists a short exact sequence  $0 \rightarrow K \rightarrow M \rightarrow M' \rightarrow 0$ , where  $M'$  is an  $S$ -module of length  $h$ . The long exact sequence in Ext gives:

$$\cdots \rightarrow \text{Ext}_S^d(M', H_I^{n-d}(S)) \rightarrow \text{Ext}_S^d(M, H_I^{n-d}(S)) \rightarrow \text{Ext}_S^d(K, H_I^{n-d}(S)) \rightarrow \cdots$$

Since  $\text{Ext}_S^d(K, H_I^{n-d}(S)) = \text{Ext}_S^d(M', H_I^{n-d}(S)) = 0$  by the inductive hypothesis,  $\text{Ext}_S^d(M, H_I^{n-d+1}(S)) = 0$ , and the claim follows.

This claim implies that  $\text{Ext}_S^d(S/\eta^\ell, H_I^{n-d}(S)) = 0$  for all  $\ell \geq 1$ . Then  $H_\eta^d H_I^{n-d}(S) = \varinjlim_\ell \text{Ext}_S^d(S/\eta^\ell, H_I^{n-d}(S)) = 0$ , the sought contradiction.  $\square$

**Proposition 3.10.** *Let  $(V, pV, K)$  be an complete DVR of unramified mixed characteristic  $p > 0$ , and let  $S = V[[x_1, \dots, x_n]]$ . Let  $f_1, \dots, f_\ell \in S$  be a regular sequence. Then*

$$\tilde{\lambda}_{i,j}(S/(f_1, \dots, f_\ell)) = 1$$

*for  $i = j = n + 1 - \ell$ , and vanishes otherwise.*

*Proof.* Our proof will be by induction on  $\ell$ . If  $\ell = 1$ , we have the short exact sequence

$$0 \rightarrow S \rightarrow S_{f_1} \rightarrow H_{f_1 S}^1(S) \rightarrow 0.$$

Then  $\text{Ext}_S^i(K, S) \cong \text{Ext}_S^{i+1}(K, H_{f_1 S}^1(S))$  for every  $i \geq 0$  because  $\text{Ext}_S^i(K, S_f) = 0$ .

Suppose that the formula holds for  $\ell - 1$  and we will prove it for  $\ell$ . From the exact sequence

$$0 \rightarrow H_{(f_1, \dots, f_{\ell-1})S}^{n-\ell-1}(S) \rightarrow H_{(f_1, \dots, f_{\ell-1})S}^{n-\ell-1}(S_{f_\ell}) \rightarrow H_{(f_1, \dots, f_\ell)S}^{n-\ell}(S) \rightarrow 0,$$



we obtain that  $\text{Ext}_S^i(K, H_{(f_1, \dots, f_\ell)S}^{n-\ell}(S)) = \text{Ext}_S^{i+1}(K, H_{(f_1, \dots, f_{\ell-1})S}^{n-\ell-1}(S))$  for every  $i \geq 0$  because  $\text{Ext}_S^i(K, H_{(f_1, \dots, f_\ell)S}^{n-\ell}(S_{f_{\ell+1}})) = 0$ .  $\square$

#### 4. EXISTENCE OF THE HIGHEST LYUBEZNIK NUMBER IN MIXED CHARACTERISTIC

**Lemma 4.1.** *Let  $(V, pV, K)$  be a complete DVR of unramified mixed characteristic  $p > 0$ , and let  $S = V[[x_1, \dots, x_n]]$ . Then*

$$\text{End}_{D(S,V)}(E_S(K)) = V.$$

*Proof.* Let  $\phi \in \text{End}_{D(S,V)}(E_S(K)) \subseteq \text{End}_S(E_S(K)) = S$ ;  $\phi$  must correspond to multiplication by some  $r \in S$ . Thus,  $\partial(rw) = r\partial(w)$  for every  $w \in E_S(K)$  and  $\partial \in D(S, V)$ . We will prove that  $r \in V$  by contradiction. If  $r \notin V$ , we may assume there exists  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n \setminus \{(0, \dots, 0)\}$  such that

$$r = a + bx^\alpha + \sum_{\beta \in \mathbb{N}, \beta \geq_{\text{lex}} \alpha} c_\beta x^\beta,$$

where  $a, b, c_\beta \in V$  and  $b \neq 0$ . Then for every  $j \in \mathbb{N}$ ,

$$\begin{aligned} r \frac{(-1)^{\alpha_1-1}}{\alpha_1!} \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{(-1)^{\alpha_n-1}}{\alpha_n!} \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} \frac{1}{p^j x_1 \cdots x_n} &= \frac{r}{p^j x_1^{\alpha_1+1} \cdots x_n^{\alpha_n+1}} \\ &= \frac{a}{p^j x_1^{\alpha_1+1} \cdots x_n^{\alpha_n+1}} + \frac{b}{p^j x_1 \cdots x_n} \end{aligned}$$

On the other hand, for every  $j \in \mathbb{N}$ ,

$$\begin{aligned} &\frac{(-1)^{\alpha_1-1}}{\alpha_1!} \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{(-1)^{\alpha_n-1}}{\alpha_n!} \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} \frac{r}{p^j x_1 \cdots x_n} \\ &= \frac{(-1)^{\alpha_1-1}}{\alpha_1!} \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{(-1)^{\alpha_n-1}}{\alpha_n!} \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} \frac{a}{p^j x_1 \cdots x_n} \\ &= \frac{a}{p^j x_1^{\alpha_1+1} \cdots x_n^{\alpha_n+1}} \end{aligned}$$

Then

$$\frac{a}{p^j x_1^{\alpha_1+1} \cdots x_n^{\alpha_n+1}} + \frac{b}{p^j x_1 \cdots x_n} = \frac{a}{p^j x_1^{\alpha_1+1} \cdots x_n^{\alpha_n+1}},$$

so  $b \in p^j V$  for every  $j \in \mathbb{N}$ . This means that  $b = 0$ , a contradiction. Thus,  $r \in V$ . Since every map given by multiplication by an element in  $V$  is already a map of  $D(S, V)$ -modules, we are done.  $\square$

**Proposition 4.2.** *Let  $(V, pV, K)$  be a complete DVR of unramified mixed characteristic  $p > 0$ , and let  $S = V[[x_1, \dots, x_n]]$ . Let  $N \subsetneq E_S(K)$  be a proper  $D(S, V)$ -submodule. Then  $N = \text{Ann}_{E_S(K)} p^\ell S$  for some  $\ell \in \mathbb{N}$ .*

*Proof.* Let  $v \in N$  be such that  $v \in \text{Ann}_{E_S(K)} p^\ell S$  but  $v \notin \text{Ann}_{E_S(K)} p^{\ell-1} S$ . We claim that  $D(S, V) \cdot v = \text{Ann}_{E_S(K)} p^\ell S$  by induction on  $\ell$ . If  $\ell = 1$ ,  $\text{Ann}_{E_S(K)} pS = E_{S/pS}(K)$ , a simple  $D(S, K)$ -module, and the claim holds. Now, we suppose the claim true for  $\ell - 1$ . Since  $\text{Ann}_{E_S(K)} p^\ell S / \text{Ann}_{E_S(K)} p^{\ell-1} S = E_{S/pS}(K)$ , there exists an operator  $\partial \in D(S, V)$  such that

$$\partial v = 1/p^\ell x_1 \cdots x_n + w$$

for an element  $w \in \text{Ann}_{E_S(K)} p^{\ell-1}S$ . Then  $p\delta v \in \text{Ann}_{E_S(K)} p^{\ell-1}S \setminus \text{Ann}_{E_S(K)} p^{\ell-2}S$ . Thus, there exists an operator  $\delta$  such that  $p\delta\partial v = w$  by the induction hypothesis, so  $1/p^\ell x_1 \cdots x_n = (\partial - p\delta\partial)v$ . Therefore,  $\text{Ann}_{E_S(K)} p^\ell = D(S, V) \cdot 1/p^\ell x_1 \cdots x_n \subseteq D(S, V) \cdot v \subseteq \text{Ann}_{E_S(K)} p^\ell$ , proving our claim.

Since  $N \neq E_S(K)$ ,  $\ell = \inf\{j \in \mathbb{N} \mid 1/p^j x_1 \cdots x_n \in N\}$  is a natural number. Hence,  $N = D(S, V) \cdot 1/p^\ell x_1 \cdots x_n = \text{Ann}_{E_S(K)} p^\ell S$ .  $\square$

**Lemma 4.3.** *Let  $(V, pV, K)$  be a complete DVR of unramified mixed characteristic  $p > 0$ , and let  $S = V[[x_1, \dots, x_n]]$ . Let  $M \subseteq \bigoplus_{i=1}^h E_S(K)$  be a  $D(S, V)$ -submodule. Then  $M \xrightarrow{\mathcal{P}} M$  is surjective if and only if  $M$  is an injective  $S$ -module.*

*Proof.* Suppose that  $M$  is an injective  $S$ -module. Since  $E_S(K) \xrightarrow{\mathcal{P}} E_S(K)$  is surjective and  $M = \bigoplus_{\ell} E_S(K)$ ,  $M \xrightarrow{\mathcal{P}} M$  is also surjective.

Now assume that  $M \xrightarrow{\mathcal{P}} M$  is surjective. We will show that  $M$  is injective by contradiction. Suppose that  $M \neq E_S(M)$ . As  $M$  is a  $D(S, V)$ -module supported only at the maximal ideal,  $\text{inj. dim}(M) \leq 1$  (cf. [NB12, Zho98]). Let  $0 \rightarrow M \rightarrow E_1 \xrightarrow{\phi} E_2 \rightarrow 0$  be a minimal injective resolution of  $M$ ; in particular,  $E_2 \neq 0$ . Let  $\mu_i = \text{Ext}_S^i(K, M)$ . Now,  $\mu_1$  is finite and less than or equal to  $h$ . Let  $(-)^* = \text{Hom}_S(-, E_S(K))$  be the Matlis duality functor. From the short exact sequence  $0 \rightarrow E_2^* \xrightarrow{\phi^*} S^{\mu_1} \rightarrow M^* \rightarrow 0$ , we obtain that  $E_2^*$  is a free module of finite rank less than or equal to  $\mu_1$ , so,  $E_2 = \bigoplus_{i=2}^{\mu_2} E_S(K)$ . By Lemma 4.1,  $\phi$  is given by a  $\mu_1 \times \mu_2$ -matrix with entries in  $V$ . Thus,  $\phi^* : S^{\mu_2} \rightarrow S^{\mu_1}$  can be represented as a matrix by the transpose of a matrix that represents  $\phi$ . We may consider  $\phi^*$  as a map of free  $V$ -modules,  $\phi^* : V^{\mu_2} \rightarrow V^{\mu_1}$ . By the structure theorem for finitely-generated modules over a principal ideal domain, there are isomorphisms  $\varphi_1 : V^{\mu_1} \rightarrow V^{\mu_1}$  and  $\varphi_2 : V^{\mu_2} \rightarrow V^{\mu_2}$ , such that  $\varphi_1 \phi^* \varphi_2$  is a matrix whose entries are zero off the diagonal. That is, we have

$$\begin{array}{ccc} V^{\mu_2} & \xrightarrow{\phi^*} & V^{\mu_1} \\ \varphi_2 \uparrow & & \downarrow \varphi_1 \\ V^{\mu_2} & \xrightarrow{\varphi_1 \phi^* \varphi_2} & V^{\mu_1} \end{array}$$

Let  $a_1, \dots, a_{\mu_1} \in V$  be the elements on the diagonal of  $\varphi_1 \phi^* \varphi_2$ , and let  $v : V \rightarrow \mathbb{N}$  be the valuation. Since  $E_S(M) = E_1 \rightarrow E_2$  is surjective, none of  $a_1, \dots, a_{\mu_1}$  is zero. Since  $E_2 \neq 0$  and the injective resolution  $0 \rightarrow M \rightarrow E_1 \xrightarrow{\phi} E_2 \rightarrow 0$  is minimal, none of  $a_1, \dots, a_{\mu_1}$  are units in  $V$ . Then  $a_1, \dots, a_{\mu_1} \in pV \setminus \{0\}$ . We extend  $\varphi_i$  as an isomorphism of  $S$ -modules,  $\varphi_i : S^{\mu_i} \rightarrow S^{\mu_i}$ . Then  $\varphi_i^* : E_i \rightarrow E_i$  is an isomorphism of  $D(S, V)$ -modules. We obtain the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \longrightarrow & E_1 & \xrightarrow{\phi} & E_2 \longrightarrow 0 \\ & & \varphi_1^* \uparrow & & \varphi_1^* \uparrow & & \downarrow \varphi_2^* \\ 0 & \longrightarrow & \text{Ker}(\varphi_2^* \phi \varphi_1^*) & \longrightarrow & E_1 & \xrightarrow{\varphi_2^* \phi \varphi_1^*} & E_2 \longrightarrow 0 \end{array}$$



Therefore,  $M \cong \text{Ker}(\varphi_2^* \phi \varphi_1^*) = \bigoplus_{i=1}^{\mu_1 - \mu_2} E_S(K) + \bigoplus_{i=1}^{\mu_2} \text{Ann}_{E_S(K)} p^{v(a_i)} S$  which is a contradiction because

$$\bigoplus_{i=1}^{\mu_1 - \mu_2} E_S(K) + \bigoplus_{i=1}^{\mu_2} \text{Ann}_{E_S(K)} p^{v(a_i)} S \xrightarrow{\mathcal{P}} \bigoplus_{i=1}^{\mu_1 - \mu_2} E_S(K) + \bigoplus_{i=1}^{\mu_2} \text{Ann}_{E_S(K)} p^{v(a_i)} S$$

is not surjective.  $\square$

**Lemma 4.4.** *Let  $(V, pV, K)$  be a complete DVR of unramified mixed characteristic  $p > 0$ , and let  $S = V[[x_1, \dots, x_n]]$ . Let  $m$  denote the maximal ideal of  $S$ , and let  $I \subseteq S$  be an ideal such that  $\dim(S/I) = 1$ . Then  $H_m^0 H_I^n(S) = 0$  and  $H_m^1 H_I^n(S) \cong E_S(K)$ .*

*Proof.* Let  $f \in m$  be an element not in any minimal prime of  $I$ , so  $\sqrt{I + fS} = m$ . We have the following short exact sequence

$$0 \rightarrow H_I^n(S) \rightarrow H_I^n(S_f) \rightarrow H_{I+fS}^{n+1}(S) \cong E_S(K) \rightarrow 0.$$

Since  $f \in m$  and  $H_I^n(S_f) \cong H_I^n(S)_f$ ,  $H_m^0 H_I^n(S_f) = 0$ , which implies that  $H_m^0 H_I^n(S) = 0$ . Moreover,  $H_m^1 H_I^n(S) = E_S(K)$ .  $\square$

**Lemma 4.5.** *Let  $(V, pV, K)$  be a complete DVR of unramified mixed characteristic  $p > 0$ , and let  $S = V[[x_1, \dots, x_n]]$ . Let  $I \subseteq S$  be an ideal of pure dimension 2. Then  $H_I^n(S)$  is an injective  $S$ -module supported only at the maximal ideal.*

*Proof.* Let  $R$  denote  $S/pS$ . The short exact sequence  $0 \rightarrow S \xrightarrow{\mathcal{P}} S \rightarrow R \rightarrow 0$  induces the long exact sequence  $\dots \rightarrow H_I^n(S) \xrightarrow{\mathcal{P}} H_I^n(S) \rightarrow H_I^n(R) \rightarrow 0$ , where  $H_I^n(R) = 0$  by the Hartshorne-Lichtenbaum Vanishing Theorem, as  $\sqrt{I + pS} \neq m$ . Thus,  $H_I^n(S) \xrightarrow{\mathcal{P}} H_I^n(S)$  is surjective. Now,  $H_{IS_P}^n(S_P) = 0$  for every prime ideal  $P$  not containing  $I$ . If  $I \subseteq P$  and  $P \neq m$ , then  $\dim(S/P) = 1$  and  $H_{IR_P}^n(R_P) = 0$  by the Hartshorne-Lichtenbaum Vanishing Theorem because  $I$  has pure dimension 2 and  $\sqrt{IS_P} \neq PS_P$ . Therefore,  $H_I^n(R)$  is a  $D(S, V)$ -module supported only at the maximal ideal. Since  $\dim_K \text{Ext}_S^0(K, H_I^n(S))$  is finite,  $H_I^n(S)$  is injective by Lemma 4.3.  $\square$

**Lemma 4.6.** *Let  $(V, pV, K)$  be a complete DVR of unramified mixed characteristic  $p > 0$ , and let  $S = V[[x_1, \dots, x_n]]$ . Let  $m$  denote the maximal ideal of  $S$ , and let  $I \subseteq S$  be an ideal of pure dimension two. Then  $H_m^0 H_I^{n-1}(S) = H_m^1 H_I^{n-1}(S) = 0$ . Moreover,  $H_I^n(S) \cong E_S(K) \oplus^\alpha$  for some  $\alpha \in \mathbb{N}$ , and  $H_m^2 H_I^{n-1}(S) \cong E_S(K) \oplus^{\alpha+1}$ . In particular,  $H_m^2 H_I^{n-1}(S)$  is an injective  $S$ -module.*

*Proof.* Let  $f \in m$  be an element not in any minimal prime of  $I$ . Then  $\sqrt{I + fS} \neq m$ . Applying the Hartshorne-Lichtenbaum Vanishing Theorem, since  $H_I^n(S)$  is supported at  $m$  by Lemma 4.5, we obtain the exact sequence

$$0 \rightarrow H_I^{n-1}(S) \rightarrow H_I^{n-1}(S_f) \rightarrow H_{I+fS}^n(S) \rightarrow H_I^n(S) \rightarrow 0.$$

Splitting the sequence into two short exact sequences, we obtain

$$\begin{aligned} 0 \rightarrow H_I^{n-1}(S) \rightarrow H_I^{n-1}(S_f) \rightarrow M \rightarrow 0, \text{ and} \\ 0 \rightarrow M \rightarrow H_{I+fS}^n(S) \rightarrow H_I^n(S) \rightarrow 0. \end{aligned}$$

These induce the following long exact sequences:

$$\begin{array}{ccccccc}
0 & \longrightarrow & H_m^0 H_I^{n-1}(S) & \longrightarrow & H_m^0 H_I^{n-1}(S_f) & \longrightarrow & H_m^0(M) \\
& & \searrow & & \searrow & & \searrow \\
& & H_m^1 H_I^{n-1}(S) & \longrightarrow & H_m^1 H_I^{n-1}(S_f) & \longrightarrow & H_m^1(M) \\
& & \searrow & & \searrow & & \searrow \\
& & H_m^2 H_I^{n-1}(S) & \longrightarrow & H_m^2 H_I^{n-1}(S_f) & \longrightarrow & H_m^2(M) \longrightarrow 0,
\end{array}$$

and

$$\begin{array}{ccccccc}
0 & \longrightarrow & H_m^0(M) & \longrightarrow & H_m^0 H_{I+fS}^n(S) & \longrightarrow & H_m^0 H_I^n(S) \\
& & \searrow & & \searrow & & \searrow \\
& & H_m^1(M) & \longrightarrow & H_m^1 H_{I+fS}^n(S) & \longrightarrow & H_m^1 H_I^n(S) \\
& & \searrow & & \searrow & & \searrow \\
& & H_m^2(M) & \longrightarrow & H_m^2 H_{I+fS}^n(S) & \longrightarrow & H_m^2 H_I^n(S) \longrightarrow 0.
\end{array}$$

Since all  $H_m^j H_I^{n-1}(S_f) = 0$ , we know that  $H_m^0 H_I^{n-1}(S) = H_m^2(M) = 0$ . Since  $\dim(S/(I+fS)) = 1$ ,  $H_m^0 H_{I+fS}^n(S) = H_m^2 H_{I+fS}^n(S) = 0$  by Lemma 4.4, which implies both that  $H_m^0(M) = H_m^1 H_I^{n-1}(S) = 0$  and that  $H_m^1(M) \cong H_m^2 H_I^{n-1}(S)$ . In addition,  $H_m^1 H_I^n(S) = H_m^2 H_I^n(S) = 0$  by Lemma 4.5. Thus, we have a short exact sequence

$$0 \rightarrow H_m^0(H_I^n(S)) \rightarrow H_m^2 H_I^{n-1}(S) \rightarrow H_m^1 H_{I+fS}^n(S) \rightarrow 0.$$

By Lemma 4.5,  $H_I^n(S)$  is an injective  $S$ -module supported only at  $m$ , and its Bass numbers are finite by [Lyu00b, NB12], so  $H_m^0 H_I^n(S) = H_I^n(S) \cong E_R(K)^{\oplus \alpha}$  for some  $\alpha \in \mathbb{N}$ . Moreover, by Lemma 4.4,  $H_m^1 H_{I+fS}^n(S) \cong E_S(K)$ .

Thus, we have the short exact sequence

$$0 \rightarrow E_S(K)^{\oplus \alpha} \rightarrow H_m^2 H_I^{n-1}(S) \rightarrow E_S(K) \rightarrow 0,$$

which splits, so that  $H_m^2 H_I^{n-1}(S) \cong E_S(K)^{\oplus \alpha+1}$ .  $\square$

**Corollary 4.7.** *Let  $(V, pV, K)$  be a complete DVR of unramified mixed characteristic  $p > 0$ , and let  $S = V[[x_1, \dots, x_n]]$ . Let  $I$  be an ideal of  $S$  of pure dimension two. Then  $H_Q^j H_I^i(S_Q)$  is injective for every prime ideal  $Q$  of  $S$ .*

*Proof.* This follows from Lemmas 4.4 and 4.6.  $\square$

**Lemma 4.8.** *Let  $(V, pV, K)$  be a complete DVR of unramified mixed characteristic  $p > 0$ , and let  $S = V[[x_1, \dots, x_n]]$ . Let  $I$  be an ideal of  $S$  such that  $\dim(S/I) = 2$ , and let  $m$  denote its maximal ideal. Then  $H_m^0 H_I^{n-1}(S) = H_m^1 H_I^{n-1}(S) = 0$  and  $H_m^2 H_I^{n-1}(S)$  is an injective  $S$ -module.*

*Proof.* Let  $J_1$  and  $J_2$  be two ideals of pure dimensions 1 and 2, respectively, such that  $I = J_1 \cap J_2$ . Using the Mayer-Vietoris sequence of local cohomology, we obtain that  $H_I^{n-1}(S) = H_{J_2}^{n-1}(S)$ . Thus, for all  $j$ ,  $H_m^j(H_I^{n-1}(S)) = H_m^j(H_{J_2}^{n-1}(S))$ , and the result follows by Lemma 4.6.  $\square$

**Proposition 4.9.** *Let  $(V, pV, K)$  be a complete DVR of unramified mixed characteristic  $p > 0$ , and let  $S = V[[x_1, \dots, x_n]]$ . Let  $I$  be an ideal of  $S$  such that  $\dim(S/I) = d$ ,*

and let  $m$  denote its maximal ideal. Then  $H_m^d H_I^{n-d+1}(S)$  is an injective  $S$ -module, and  $H_m^j H_I^{n-d+1}(S) = 0$  for  $j > d$ .

*Proof.* We proceed by induction on  $d$ . If  $d = 0, 1$ , or  $2$ , we have the result by Lemmas 4.4 and 4.8. Suppose that  $d \geq 3$  and the statement holds for  $d-1$ . If  $\text{Ass}_S H_I^{n-d}(S) \neq \{m\}$ , we pick an element  $r \in m$  that is neither in any minimal prime of  $I$ , nor of  $H_I^{n-d}(S)$ , which is possible because  $\text{Ass}_S H_I^{n-d}(S)$  is finite (cf. [Lyu00b, NB12]). On the other hand,  $\text{Ass}_S H_I^{n-d}(S) = \{m\}$ , we pick an element  $r \in m$  not in any minimal prime of  $I$ . We have that  $H_m^d(H_I^{n-d+1}(S)) = H_m^{d-1} H_{I+rS}^{n-d+2}(S)$  and  $H_m^j H_I^{n-d+1}(S) = H_m^{j-1} H_{I+rS}^{n-d+2}(S) = 0$  for  $j > d$  as in the proof of [Zha07, Proposition 2.1] because the conclusions of in [Zha07, Lemmas 2.3 and 2.4] hold in our case. Hence, the result follows by the induction hypothesis.  $\square$

**Theorem 4.10.** *Let  $(S, m, K)$  be either a regular local ring of unramified mixed characteristic, or a regular local ring containing a field. Let  $n = \dim(S)$ , and let  $I$  be an ideal of  $S$  such that  $\dim(S/I) = d$ . Then  $\text{inj. dim } H_I^{n-d}(S) = d$ .*

*Proof.* We need to prove that  $\text{Ext}^j(R_Q/Q R_Q, H_{IR_Q}^i(R_Q)) = 0$  for every prime ideal  $Q$  of  $R$ , all  $i \in \mathbb{N}$ , and all  $j > d$ . We may assume that  $Q$  is  $m$ , the maximal ideal of  $S$ , because if  $Q \subsetneq m$ , then  $\dim R_Q/IR_Q < d$  and  $\text{inj. dim}_{R_Q} H_{IR_Q}^i(R_Q) \leq \dim_{R_Q} H_{IR_Q}^i(R_Q) \leq d$  by [Zho98, Theorem 5.1].

We proceed by induction on  $n$ . If  $n = 0$ ,  $S$  is a field and the result follows. Assume that the statement holds for all such  $S$  of dimension less than  $n$ .

Since the theorem is already true for regular local rings that contain a field (cf. [HS93, Lyu93, Lyu00c]), we will focus on the case where  $S$  is an unramified regular local ring of unramified mixed characteristic.

Let  $E^* = E^1 \rightarrow E^2 \rightarrow \dots$  be a minimal injective resolution for  $H_I^{n-d+1}(S)$ . By [Zho98, Theorem 5.1],  $E^j = 0$  for  $j > d+1$ . For every prime ideal  $Q \subseteq S$ ,  $S_Q$  is either an unramified regular local ring of mixed characteristic or a regular local ring containing a field. Moreover,  $\dim(S_Q/IS_Q) \leq d-1$  for every prime ideal  $Q \subsetneq m$ . Thus,  $(E^d)_Q = (E^{d+1})_Q = 0$  by the inductive hypothesis. Hence,  $E^d$  and  $E^{d+1}$  are supported only at  $m$ .

Let  $M = \text{Im}(E^{d-1} \rightarrow E^d) = \text{Ker}(E^d \rightarrow E^{d+1})$ . It suffices prove that  $M$  is an injective  $S$ -module. The modules  $H_m^j H_I^{n-d}(S)$  can be computed from the complex

$$H_m^0(E^*) = H_m^0(E^1) \rightarrow H_m^0(E^2) \rightarrow \dots$$

Let  $B^j = \text{Im}(H_m^0(E^{j-1}) \rightarrow H_m^0(E^j))$  and  $Z^j = \text{Ker}(H_m^0(E^j) \rightarrow H_m^0(E^{j+1}))$ . Note that  $Z^d = M$  since  $E_d$  and  $E_{d+1}$  are supported only at  $m$ . Since  $\text{inj. dim } Z^j \leq 1$  and  $\text{inj. dim } H_m^j H_I^{n-d}(S) \leq 1$  by the proof of [Zho98, Theorem 5.1] or by [NB12, Theorem 0.3], as in the proof of [Zho98, Theorem 5.1], we obtain that  $B^j$  is injective from the following short exact sequences:

$$\begin{aligned} 0 \rightarrow Z^j \rightarrow H_m^0(E^j) \rightarrow B^j \rightarrow 0, \text{ and} \\ 0 \rightarrow B^{j-1} \rightarrow Z^j \rightarrow H_m^j(H_I^{n-d}(S)) \rightarrow 0. \end{aligned}$$

Since  $H_m^d H_I^{n-d}(S)$  injective by Proposition 4.9, we know that  $Z^d = M$  is injective due to the short exact sequence  $0 \rightarrow B^{d-1} \rightarrow Z^d \rightarrow H_m^d H_I^{n-d}(S) \rightarrow 0$ . Therefore,  $E_{d+1} = 0$ , so  $\text{inj. dim } H_I^{n-d}(S) = d$ .  $\square$

**Definition 4.11** (Highest Lyubeznik number in mixed characteristic). For  $(R, m, K)$  a local ring of dimension  $d$  such that  $\text{char}(K) = p > 0$ , the *highest Lyubeznik number in mixed characteristic of  $R$*  is defined as  $\tilde{\lambda}_{d,d}(R)$ .

Note that the nomenclature “highest” is justified since  $\tilde{\lambda}_{i,d}(R) = 0$  for  $i > d$  by Theorem 4.10. Moreover, we may also justify the following definition:

**Definition 4.12** (Lyubeznik table in mixed characteristic). For  $(R, m, K)$  a local ring such that  $\text{char}(K) = p > 0$  and  $d = \dim(R)$ , the *Lyubeznik table in mixed characteristic* is the  $(d+1) \times (d+1)$  matrix  $\tilde{\Lambda}(R)$ , where  $\tilde{\Lambda}(R)_{i,j} = \tilde{\lambda}_{i,j}(R)$  for  $0 \leq i, j \leq d$ .

**Remark 4.13.** Recall that for a local ring  $R$  of dimension  $d$  containing a field, the *Lyubeznik table* of  $R$  is defined as the  $(d+1) \times (d+1)$  matrix  $\tilde{\Lambda}(R)$  such that  $\tilde{\Lambda}(R)_{i,j} = \lambda_{i,j}(R)$  for  $0 \leq i, j \leq d$ . This matrix contains all nonzero Lyubeznik numbers, and is also upper triangular, since  $\lambda_{i,j}(R) = 0$  if either  $i > j$  or  $j > d$  [Lyu93, Properties 4.4i, 4.4ii].

On the other hand, Proposition 3.9 and Theorem 4.10 imply that the Lyubeznik table in mixed characteristic contain all nonzero Lyubeznik numbers in mixed characteristic. However, Proposition 3.9 only implies that the Lyubeznik table in mixed characteristic is nonzero below the subdiagonal.

## 5. EXAMPLES IN CHARACTERISTIC $p > 0$ WHERE THE EQUAL-CHARACTERISTIC AND THE LYUBEZNIK NUMBERS IN MIXED CHARACTERISTIC ARE EQUAL

**Lemma 5.1.** *Let  $(V, pV, K)$  be a complete DVR of unramified mixed characteristic  $p > 0$ , and let  $S = V[[x_1, \dots, x_n]]$ . Let  $M$  be an  $S$ -module such that  $\dim_K \text{Ext}_S^i(K, M)$  is finite for all  $i \in \mathbb{N}$ . Suppose that  $M \xrightarrow{p} M$  is surjective. Then for all  $i \in \mathbb{N}$ ,*

$$\dim_K \text{Ext}_S^i(K, M) = \dim_K \text{Ext}_{S/pS}^i(K, \text{Ann}_M pS).$$

*Proof.* Let  $R = S/pS$  and  $N = \text{Ann}_M(pS)$ . The short exact sequence  $0 \rightarrow N \rightarrow M \xrightarrow{p} M \rightarrow 0$  induces a long exact sequence

$$0 \rightarrow \text{Ext}_S^0(K, N) \rightarrow \text{Ext}_S^0(K, M) \xrightarrow{p} \text{Ext}_S^0(K, M) \rightarrow \text{Ext}_S^1(K, N) \rightarrow \dots$$

Since multiplication by  $p$  is zero on  $\text{Ext}_S^i(K, M)$ , we have short exact sequences

$$0 \rightarrow \text{Ext}_S^{i-1}(K, M) \rightarrow \text{Ext}_S^i(K, N) \rightarrow \text{Ext}_S^i(K, M) \rightarrow 0.$$

for all  $i \in \mathbb{N}$ . Thus,

$$\dim_K \text{Ext}_S^i(K, N) = \dim_K \text{Ext}_S^{i-1}(K, M) + \dim_K \text{Ext}_S^i(K, M).$$

We can compute  $\text{Ext}_S^i(K, N)$  using the Koszul complex,  $\mathcal{K}$ , with respect to the sequence  $p, x_1, \dots, x_n$  in  $S$ . On the other hand, we can compute  $\text{Ext}_R^i(K, N)$  using the Koszul complex,  $\overline{\mathcal{K}}$ , with respect to the sequence  $\overline{x}_1, \dots, \overline{x}_n$ , in  $R$ . Now,  $\mathcal{K}(N)$  is the direct sum of  $\overline{\mathcal{K}}(N)$  and an indexing shift of the same complex by one. This means that

$$\dim_K \text{Ext}_S^i(K, N) = \dim_K \text{Ext}_R^{i-1}(K, N) + \dim_K \text{Ext}_R^i(K, N), \text{ so}$$

$$\dim_K \text{Ext}_S^{i-1}(K, M) + \dim_K \text{Ext}_S^i(K, M) = \dim_K \text{Ext}_R^{i-1}(K, N) + \dim_K \text{Ext}_R^i(K, N).$$

Since  $\dim_K \operatorname{Ext}_S^{-1}(K, M) = \dim_K \operatorname{Ext}_R^{-1}(K, N) = 0$ , we know that  $\dim_K \operatorname{Ext}_S^0(K, M) = \dim_K \operatorname{Ext}_R^0(K, N)$  as well. Inductively,  $\dim_K \operatorname{Ext}_S^i(K, M) = \dim_K \operatorname{Ext}_R^i(K, N)$  for all  $i \in \mathbb{N}$ .  $\square$

**Corollary 5.2.** *Let  $(V, pV, K)$  be a complete DVR of unramified mixed characteristic  $p > 0$ , and let  $S = V[[x_1, \dots, x_n]]$ . Let  $I$  be an ideal of  $S$  such that  $S/I$  is a Cohen-Macaulay ring of characteristic  $p$ . Then for all  $i, j \in \mathbb{N}$ ,*

$$\dim_K \operatorname{Ext}_{S/pS}^i(K, H_{IS/pS}^{n-j}(S/pS)) = \dim_K \operatorname{Ext}_S^i(K, H_I^{n+1-j}(S)).$$

*Proof.* Let  $R = S/pS$ . The short exact sequence  $0 \rightarrow S \xrightarrow{p} S \rightarrow R \rightarrow 0$  induces the short exact sequence

$$0 \rightarrow H_I^{n-d}(R) \rightarrow H_I^{n-d+1}(S) \xrightarrow{p} H_I^{n-d+1}(S) \rightarrow 0$$

since  $H_I^{n-d+1}(R) = 0$  by [PS73, Proposition 4.1]. Since  $H_I^{n-d}(S) = 0$   $H_I^i(S) \xrightarrow{p} H_I^i(S)$  is injective for  $i \neq n-d+1$ , and  $H_I^{n-d}(S) = 0$ . The result then follows from Lemma 5.1.  $\square$

**Proposition 5.3.** *Let  $(V, pV, K)$  be a complete DVR of unramified mixed characteristic  $p > 0$ , and let  $S = V[[x_1, \dots, x_n]]$ . Let  $I$  be an ideal of  $S$  containing  $p$ , such that  $\dim(S/I) \leq 2$ . Then*

$$\dim_K \operatorname{Ext}_{S/pS}^d(K, H_{IS/pS}^{n-d}(S/pS)) = \dim_K \operatorname{Ext}_S^d(K, H_I^{n+1-d}(S)).$$

*Proof.* Let  $R = S/pS$ . Consider the following cases.

If  $\dim(S/I) = 0$ ,  $H_I^{n+1}(S) = E_S(K)$  and  $H_I^n(S) = E_R(K)$ . Then

$$\dim_K \operatorname{Ext}_{S/pS}^d(K, H_{IS/pS}^{n-d}(S/pS)) = \dim_K \operatorname{Ext}_S^d(K, H_I^{n+1-d}(S)) = 1.$$

If  $\dim(S/I) = 1$ , the short exact sequence  $0 \rightarrow S \xrightarrow{p} S \rightarrow R \rightarrow 0$  induces a long exact sequence  $0 \rightarrow H_I^{n-1}(R) \rightarrow H_I^n(S) \xrightarrow{p} H_I^n(S) \rightarrow 0$  by the Hartshorne-Lichtenbaum vanishing theorem. The proposition then follows from Lemma 5.1.

Suppose that  $\dim(S/I) = 2$ . First assume that  $I$  has pure dimension 2. Let  $\alpha$  be the number of connected components of  $\operatorname{Spec}(\widehat{A}) \setminus \{m\}$ , where  $A = \widehat{R/I}^{sh}$  is the strict Henselization of  $R/I$ . In fact,  $\alpha = \dim_K \operatorname{Ext}_R^2(K, H_I^{n-2}(S/pS))$  (cf. [Wal01, Proposition 2.2]).

We prove the statement by induction on  $\alpha$ . If  $\alpha = 1$ , the short exact sequence  $0 \rightarrow S \xrightarrow{p} S \rightarrow R \rightarrow 0$  induces the short exact sequence

$$0 \rightarrow H_I^{n-2}(R) \rightarrow H_I^{n-1}(S) \xrightarrow{p} H_I^{n-1}(S) \rightarrow 0,$$

since  $H_I^{n-1}(R) = 0$  by [HL90, Theorem 2.9]. The proposition then follows from Lemma 5.1. If  $\alpha > 1$ , we pick ideals  $J_1, \dots, J_\alpha$  such that  $I = J_1 \cap \dots \cap J_\alpha$ , and each  $J_k$  defines a connected component of  $\operatorname{Spec}(\widehat{A}) \setminus \{m\}$ . Let  $J$  denote  $J_1 \cap \dots \cap J_{\alpha-1}$ . Using the Mayer-Vietoris sequence, we obtain an isomorphism

$$H_J^{n-1}(S) \oplus H_{J_\alpha}^{n-1}(S) \cong H_I^{n-1}(S)$$

because  $\sqrt{J + J_\alpha} = m$ . Then

$$\dim_K \operatorname{Ext}_S^2(K, H_I^{n-1}(S)) = \dim_K \operatorname{Ext}_S^2(K, H_J^{n-1}(S)) + \dim_K \operatorname{Ext}_S^2(K, H_{J_\alpha}^{n-1}(S)) = \alpha.$$

By Lemma 4.6, [Lyu93, Lemma 1.4] and [Wal01, Proposition 2.2], the other numbers are determined by  $\alpha$ .

For the general case such that  $\dim(S/I) = 2$ , let  $P_1, \dots, P_r$  be the minimal primes of dimension one of  $I$ , and let  $Q_1, \dots, Q_s$  be the minimal primes of dimension two of  $I$ . Let  $J_1 = P_1 \cap \dots \cap P_r$  and  $J_2 = Q_1 \cap \dots \cap Q_s$ . We claim that  $\text{Ext}_S^j(K, H_I^{n-1}(S)) = \text{Ext}_S^j(K, H_{J_2}^{n-1}(S))$ . Let  $f_1, \dots, f_\ell \in J_2 \setminus I$  such that  $I + (f_1, \dots, f_\ell)S = J_2$ . We proceed by induction on  $\ell$ ; first assume that  $\ell = 1$ . Since  $H_I^{n-1}(S) = H_{J_2}^{n-1}(S)$ ,  $H_I^{n-1}(S_{f_1}) = 0$ . The long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{I+f_1S}^{n-1}(S) & \longrightarrow & H_I^{n-1}(S) & \longrightarrow & H_I^{n-1}(S_{f_1}) \\ & & & & & & \searrow \\ & & & & & & \text{-----} \\ & & & & & & \nearrow \\ & & & & & & \text{-----} \\ & & & & & & \searrow \\ & & & & & & H_{I+f_1S}^n(S) \longrightarrow H_I^n(S) \longrightarrow H_I^n(S_{f_1}) \longrightarrow 0, \end{array}$$

then indicates both that  $H_{I+f_1S}^{n-1}(S) \cong H_I^{n-1}(S)$ , and that

$$0 \rightarrow H_{I+f_1S}^n(S) \rightarrow H_I^n(S) \rightarrow H_I^n(S_f) \rightarrow 0$$

is exact. Hence,  $\text{Ext}_S^j(K, H_I^{n-1}(S)) = \text{Ext}_S^j(K, H_{I+f_1S}^{n-1}(S))$ . Moreover,  $I + f_1S \subseteq J_2$  is an ideal of dimension 2, whose minimal primes of dimension 2 are  $P_1, \dots, P_r$ . If we assume that the claim is true for  $\ell$ , the proof for  $\ell + 1$  is analogous to the previous part.  $\square$

**Corollary 5.4.** *Let  $(R, m, K)$  be a local ring of characteristic  $p > 0$ . If  $R$  is a Cohen-Macaulay ring or if  $\dim R \leq 2$ , then for  $i, j \in \mathbb{N}$ ,  $\tilde{\lambda}_{i,j}(R) = \lambda_{i,j}(R)$ .*

*Proof.* Since dimension, Cohen-Macaulayness, and both Lyubeznik numbers are preserved after completion, we can assume that  $R$  is complete. Then the result follows from Corollary 5.2 and Proposition 5.3.  $\square$

## 6. AN EXAMPLE FOR WHICH THE EQUAL-CHARACTERISTIC AND THE LYUBEZNIK NUMBERS IN MIXED CHARACTERISTIC DIFFER

Throughout this section, we will often refer to the following ring and ideal:

**Notation 6.1.** Let  $R = \mathbb{Z}_Q[x_1, \dots, x_6]$ , where  $p = 2$  and  $Q = p\mathbb{Z}$ . Moreover, let  $I$  denote the ideal of  $R$  generated by the 11 elements

$$\begin{aligned} & p, x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_1x_4x_6, x_1x_5x_6, \\ & x_2x_3x_6, x_2x_4x_5, x_2x_5x_6, x_3x_4x_5, x_3x_4x_6. \end{aligned}$$

**Remark 6.2.** It is easily checked that for  $I \subseteq R$  as in Notation 6.1,  $\text{depth}_I(R) = 4$ . Thus, the short exact sequence  $0 \rightarrow R \rightarrow R_p \rightarrow R_p/R \rightarrow 0$  induces the long exact sequence

$$(6.2.1) \quad 0 \rightarrow H_I^3(R_p/R) \rightarrow H_I^4(R) \rightarrow H_I^4(R_p) \rightarrow H_I^4(R_p/R) \rightarrow \dots$$

Since  $H_I^i(R)$  is supported at  $p \in I$ , for all  $i \in \mathbb{N}$ ,  $H_I^i(R_p) = 0$ , so  $H_I^i(R_p/R) \cong H_I^{i+1}(R)$ .

**Proposition 6.3.** *With  $R, p$ , and  $I$  as in Notation 6.1, the map*

$$H_I^3(R_p/R) \xrightarrow{\mathcal{P}} H_I^3(R_p/R)$$

*is not surjective.*



*Proof.* Since  $\text{depth}_I(R) = 4$ ,  $H_I^0(R_p/R) = H_I^1(R_p/R) = H_I^2(R_p/R) = 0$  by the long exact sequence in local cohomology (see Remark 6.2). For every  $\ell \in \mathbb{N}$ , the exact sequence  $0 \rightarrow R/p^\ell R \rightarrow R_p/R \xrightarrow{p^\ell} R_p/R \rightarrow 0$  induces a long exact sequence

$$(6.3.1) \quad 0 \rightarrow H_I^3(R/p^\ell R) \rightarrow H_I^3(R_p/R) \xrightarrow{p^\ell} H_I^3(R_p/R) \xrightarrow{\partial} H_I^4(R/p^\ell R) \rightarrow \cdots.$$

Note that  $H_I^3(R/p^\ell R) = \text{Ann}_{H_I^3(R_p/R)}(p^\ell R)$ .

As the direct limit functor is exact, the limit of the direct system of short exact sequences in Figure 1 is the short exact sequence  $0 \rightarrow R/pR \rightarrow R_p/R \xrightarrow{p} R_p/R \rightarrow 0$ . Moreover,  $H_I^j(R_p/R) = \varinjlim H_I^j(R/p^\ell R)$ .

$$\begin{array}{ccccccc} 0 & \longrightarrow & R/pR & \xrightarrow{\cdot p} & R/p^2R & \longrightarrow & R/pR \longrightarrow 0 \\ & & \downarrow = & & \downarrow \cdot p & & \downarrow \cdot p \\ 0 & \longrightarrow & R/pR & \xrightarrow{\cdot p^2} & R/p^3R & \longrightarrow & R/p^2R \longrightarrow 0 \\ & & \downarrow = & & \downarrow \cdot p & & \downarrow \cdot p \\ 0 & \longrightarrow & R/pR & \xrightarrow{\cdot p^3} & R/p^4R & \longrightarrow & R/p^3R \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \vdots & & \vdots & & \vdots \end{array}$$

FIGURE 1.

By [SW11, Example 5.10], the Bockstein homomorphism  $H_I^3(R/p^\ell R) \rightarrow H_I^4(R/p^\ell R)$  is nonzero, so  $H_I^3(R/p^2R) \xrightarrow{\pi} H_I^3(R/pR)$  is not surjective by the isomorphism of sequences given in Figure 2. Therefore,  $\text{Ann}_{H_I^3(R_p/R)} p^2R \xrightarrow{p} \text{Ann}_{H_I^3(R_p/R)} pR$  is not surjective, so that  $H_I^3(R_p/R) \xrightarrow{p} H_I^3(R_p/R)$  is also not surjective.

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_I^3(R/pR) & \xrightarrow{\cdot p} & H_I^3(R/p^2R) & \xrightarrow{\pi} & H_I^3(R/p^2R) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Ann}_{H_I^3(R_p/R)} pR & \longrightarrow & \text{Ann}_{H_I^3(R_p/R)} p^2R & \xrightarrow{\cdot p} & \text{Ann}_{H_I^3(R_p/R)} pR \longrightarrow 0 \end{array}$$

FIGURE 2.

□

**Remark 6.4.** Let  $R = \mathbb{F}_2[y_1, \dots, y_5]$ , and let  $J = (y_1y_2, y_2y_3, y_3y_4, y_4y_5, y_5y_1)$ . Then  $J = (y_2, y_3, y_5) \cap (y_1, y_3, y_4) \cap (y_1, y_2, y_4) \cap (y_1, y_3, y_5) \cap (y_2, y_4, y_5)$ . Now,  $R/J$  is a graded Cohen-Macaulay ring of dimension 2, where the classes of  $y_1 + y_2 + y_3$  and  $y_1 + y_4 + y_5$  form a homogeneous system of parameters. Then  $H_I^i(R) \neq 0$  if only if  $i = 3$  [PS73, Proposition 4.1]. (See [AMGLZA03, Proposition 3.1] for an analog in characteristic zero.)

**Lemma 6.5.** *Consider  $R$ ,  $p$ , and  $I$  as in Notation 6.1. Then  $H_I^4(R/pR)$  is supported only at the maximal ideal  $(p, x_1, \dots, x_6)$ .*

*Proof.* Let  $\overline{R}$  denote  $R/pR$ . Every associated prime in  $\text{Ass}_S H_I^4(\overline{R})$  has the form  $(2, x_{i_1}, \dots, x_{i_j})R$  for some  $\{i_1, \dots, i_j\} \subseteq \{1, \dots, 6\}$  by [Yan00, Proposition 2.5] and [Yan01, Proposition 2.7], as  $I\overline{R}$  is a square-free monomial ideal. Then it suffices to prove that  $H_I^4(\overline{R})_{x_i} = 0$  for every  $1 \leq i \leq 6$ .

We first check that  $H_I^4(\overline{R})_{x_6} = 0$ . Let  $A = \mathbb{F}_2[x_1, \dots, x_5]$ , and

$$J = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1) \subseteq A.$$

Now,  $A$  and  $J$  are as in Remark 6.4, so that  $H_J^4(A) = 0$ . Since  $A[x_6]_{x_6} = \overline{R}_{x_6}$  is a flat extension,  $H_J^4(\overline{R})_{x_6} = H_J^4(A) \otimes_A \overline{R}_{x_6} = 0$ . Note that  $J\overline{R}_{x_6} = I\overline{R}_{x_6}$ , and so  $H_I^4(\overline{R})_{x_6} = H_J^4(\overline{R})_{x_6} = 0$ . The proof that  $H_I^4(\overline{R})_{x_i} = 0$  for  $1 \leq i \leq 5$  is analogous, and again relies on Remark 6.4.  $\square$

**Corollary 6.6.** *Take  $R$ ,  $p$ , and  $I$  as in Notation 6.1. Let  $S = \widehat{R}_m$ , where  $m$  is the maximal ideal  $(p, x_1, \dots, x_6)$  of  $R$ . Then*

$$\text{Coker} \left( H_I^3(S_p/S) \xrightarrow{\mathcal{P}} H_I^3(S_p/S) \right) \cong H_I^4(S/pS) \cong E_{S/pS}(\mathbb{F}_2).$$

*Proof.* Note that  $S = \widehat{\mathbb{Z}}_Q[[x_1, \dots, x_6]]$ , the power series ring with coefficients in the  $p$ -adic integers. Then  $S/pS \cong \mathbb{F}_2[[x_1, \dots, x_6]]$ , and  $H_I^4(S/pS)$  is a  $D(S/pS, \mathbb{F}_2)$ -module supported only at  $m$  by Lemma 6.5. Thus,  $H_I^4(S/pS) = E_{S/pS}(\mathbb{F}_2)$  by [AMV, Example 4.6]. Since  $\text{Coker} \left( H_I^3(S_p/S) \xrightarrow{\mathcal{P}} H_I^3(S/pS) \right)$  is nonzero by Proposition 6.7, and injects into  $H_I^4(S/pS) = E_{S/pS}(\mathbb{F}_2)$  by the long exact sequence

$$0 \rightarrow H_I^3(S/pS) \rightarrow H_I^3(S_p/S) \xrightarrow{\mathcal{P}} H_I^3(S_p/S) \rightarrow H_I^4(S/pS) \rightarrow \dots,$$

we know that  $\text{Coker} \left( H_I^3(S_p/S) \xrightarrow{\mathcal{P}} H_I^3(S/pS) \right)$  is a nonzero  $D(S/pS, \mathbb{F}_2)$ -submodule of  $E_{S/pS}(\mathbb{F}_2)$ . Hence,

$$\text{Coker} \left( H_I^3(S_p/S) \xrightarrow{\mathcal{P}} H_I^3(S_p/S) \right) \cong H_I^4(S/pS) \cong E_{S/pS}(\mathbb{F}_2),$$

since  $E_{S/pS}(\mathbb{F}_2)$  is a simple  $D(S/pS, \mathbb{F}_2)$ -module.  $\square$

**Corollary 6.7.** *There exists a regular local ring  $S$  of unramified mixed characteristic  $p > 0$ , with an ideal  $I$  of  $S$  containing  $p$ , so that the map*

$$H_I^4(S) \xrightarrow{\mathcal{P}} H_I^4(S)$$

*is not surjective.*

*Proof.* Again, consider  $R$ ,  $p$ , and  $I$  as in Notation 6.1. Let  $S = \widehat{R}_m$ , where  $m = (p, x_1, \dots, x_6)R$ . Then by Corollary 6.6,  $\text{Coker} \left( H_I^4(S) \xrightarrow{\mathcal{P}} H_I^4(S) \right) \cong E_{S/pS}(\mathbb{F}_2) \neq 0$ .  $\square$

**Proposition 6.8.** *Take  $R$ ,  $p$ , and  $I$  as in Notation 6.1. Let  $S = \widehat{R}_m$ , where  $m = (p, x_1, \dots, x_6)$ . Then*

$$\widetilde{\Lambda}(S/IS) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

*Proof.* We will abuse notation, using “ $I$ ” to mean  $IS$ . By Corollary 6.6 and Proposition 6.3, we have the short exact sequence

$$0 \rightarrow H_I^3(S/pS) \rightarrow H_I^4(S) \xrightarrow{p} H_I^4(S) \rightarrow H_I^4(S/pS) \rightarrow 0.$$

We have that  $H_I^4(S) \neq 0$  and all other  $H_I^i(S)$  must vanish, since the only way that  $H_I^i(S) \xrightarrow{p} H_I^i(S)$  is injective is if  $H_I^i(S) = 0$ . Since  $H_i^j(S) \neq 0$  if and only if  $j = 4$ , the spectral sequence

$$E_2^{p,q} = H_m^p(H_I^q(S)) \implies H_m^{p+q}(S) = E_\infty^{p,q}$$

converges at the second stage, so that  $H_m^3 H_I^4(S) \cong E_S(K)$ , and all other  $H_m^j H_I^i(S)$  vanish. Since all  $H_m^j H_I^i(S)$  are injective  $S$ -modules,  $H_m^j H_I^i(S) \cong E_S(K) \oplus \lambda_{i,j}(S)$  as in the proof of [Lyu93, Lemma 1.4], and the result follows.  $\square$

**Remark 6.9.** Together, Proposition 6.8 and [ÀMV, Example 4.6] show that the Lyubeznik numbers in mixed characteristic are not always the same as the equal characteristic ones. Take  $I \subseteq R$  as defined in Notation 6.1. Let  $S_1 = \mathbb{F}_2[[x_1, \dots, x_6]]$ , and  $S_2 = \widehat{\mathbb{Z}}_Q[[x_1, \dots, x_6]]$ , where  $Q = 2\mathbb{Z}$ . By [ÀMV, Example 4.6], the Lyubeznik numbers in equal characteristic 2 are given by

$$\Lambda(S_1/IS_1) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

On the other hand, by Proposition 6.8, the Lyubeznik numbers in mixed characteristic in in mixed characteristic 2,

$$\widetilde{\Lambda}(S_2/IS_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In particular, this is an example of a negative answer for Question 3.8.

The computation in Remark 6.9 is related to work in [ÀMGLZA03].

**Theorem 6.10.** *There exists a regular local ring  $S$  of unramified mixed characteristic  $p > 0$ , and an ideal  $I$  of  $S$ , such that for some  $i, j \in \mathbb{N}$ ,*

$$\dim_K \operatorname{Ext}_S^j(K, H_I^i(S)) \neq \dim_K \operatorname{Ext}_{S/pS}^j(K, H_{IS/pS}^{i-1}(S/pS)).$$

*Proof.* Take  $R$ ,  $p$ , and  $I$  as in Notation 6.1. Let  $S$  denote  $\widehat{R}_m$ , where  $m = (p, x_1, \dots, x_6)$ . Then  $\dim_K \operatorname{Ext}_R^0(K, H_I^5(S)) = 0 \neq 1 = \dim_K \operatorname{Ext}_{S/pS}^0(K, H_{IS/pS}^4(S/pS))$ .  $\square$

## 7. FURTHER QUESTIONS

**Question 7.1.** If  $R$  is a  $d$ -dimensional Cohen-Macaulay local ring containing a field, Kawasaki showed that  $\lambda_{d,d}(R) = 1$  [Kaw02]. If  $R$  is a  $d$ -dimensional Cohen-Macaulay local ring of mixed characteristic, is  $\tilde{\lambda}_{d,d}(R) = 1$ ?

**Question 7.2.** Does there exist a local ring  $R$  of equal characteristic  $p > 0$  such that for some  $i, j \in \mathbb{N}$ ,  $\lambda_{i,j}(R) \neq \tilde{\lambda}_{i,j}(R)$ , and both are nonzero (cf. Remark 6.9, Theorem 6.10)?

**Question 7.3.** Consider a local ring  $R$  of characteristic  $p > 0$  such that  $\lambda_{i,j}(R) \neq \tilde{\lambda}_{i,j}(R)$  for some  $i, j \in \mathbb{N}$ . The vanishing of each invariant in Remark 6.9 suggests that the equal-characteristic invariants might capture some finer information about  $R$  than do the Lyubeznik numbers in mixed characteristic. How can we characterize this data? On the other hand, do the Lyubeznik numbers in mixed characteristic capture properties of  $R$  that the equal-characteristic Lyubeznik numbers miss?

**Question 7.4.** Do the Lyubeznik numbers in mixed characteristic have geometric interpretations?

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